Tutorial 10: Selected problems of Assignment 9

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Recall the notion of Initial Value Problem :

Def An Initial Value Problem (IVP) consists of the following equations

$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where $f : R := \underbrace{[t_0-a, t_0+a]}_{I_a(t_0)} \times \underbrace{[x_0-b, x_0+b]}_{I_b(x_0)} \to \mathbb{R}$ is continuous.

An IVP is uniquely solvable for $a' \in (0,a)$ if there exists a unique function

$x(t) : I_{a'}(t_0) \to I_b(x_0)$ such that $x(t)$ is C^1 and solves IVP:

$$\begin{cases} x'(t) = f(t, x(t)), \quad \forall\, t \in I_{a'}(t_0) \\ x(t_0) = x_0 \end{cases}$$

Thm (Picard-Lindelöf) Given an IVP as above,

① If f satisfies a Lipschitz condition (uniform in t), i.e. $\exists\, L > 0$

such that $\forall\, (t,x_1), (t,x_2) \in R$, $|f(t,x_1) - f(t,x_2)| \leq L\,|x_1 - x_2|$, then

IVP is uniquely solvable for any $a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}$, $M := \sup_R |f(t,x)|$ (assuming $M>0$)

② If in addition $f \in C^k(R)$, $\exists\, k \geq 1$, then $x(t) \in C^{k+1}(I_{a'}(t_0))$

Q1) (HW9, Q4) Using the perturbation of identity, prove ①

for any $a' < \min\{a, \frac{b}{M_0+Lb}, \frac{1}{L}\}$, where $M_0 := \sup_{t\in I_a(t_0)} |f(t,x_0)|$

Pf) Recall that by Lecture note Prop. 3.11, it suffices to solve the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \text{ where } x(t): I_{a'}(t_0) \to I_b(x_0) \text{ is continuous.}$$

Equivalently: $x(t) + \left(-x_0 - \int_{t_0}^{t} (f(s,x(s)) - f(s,x_0))ds\right) = \int_{t_0}^{t} f(s,x_0)ds$

Applying the perturbation of identity with $(X, \|\cdot\|) = (C[t_0-a', t_0+a'], \|\cdot\|_\infty)$

to $\Phi : X \to X$, where $\Phi(x(t)) = x(t) + \left(-x_0 - \int_{t_0}^{t} (f(s,x(s)) - f(s,x_0))ds\right)$

$= (I + \Psi)(x(t))$, where $\Psi(x(t)) = -x_0 - \int_{t_0}^{t} (f(s,x(s)) - f(s,x_0))ds$

Let $x_0(t), y_0(t) \in X$ be defined as $\begin{cases} x_0(t) = x_0, & \forall t \in I_{a'}(t_0) \\ y_0(t) = 0 \end{cases}$

then $\Phi(x_0(t)) = y_0(t)$.

Checking Ψ is a contraction: $\forall x_1(t), x_2(t) \in X$, $\forall t \in I_{a'}(t_0)$

$$|\Psi(x_1(t)) - \Psi(x_2(t))| = \left| \int_{t_0}^{t} (f(s,x_1(s)) - f(s,x_2(s)))ds \right|$$

$$\leq \int_{t_0}^{t} L \cdot |x_1(s) - x_2(s)| ds \leq L \cdot \|x_1 - x_2\|_\infty |t - t_0| \leq (La') \cdot \|x_1 - x_2\|_\infty = \gamma \|x_1 - x_2\|_\infty$$

, where $\gamma := La' < 1$. $\therefore \|\Psi(x_1) - \Psi(x_2)\|_\infty \leq \gamma \|x_1 - x_2\|_\infty$

$\therefore$ By the perturbation of identity, choose $r = b$, $R = (1 - La')b$,

then $\forall\, y(t) \in \overline{B_R(y_0(t))}$, $\exists!\, x(t) \in B_r(x_0(t))$ such that $\Phi(x(t)) = y(t)$.

Checking $y(t) := \int_{t_0}^{t} f(s, x_0)\, ds \in B_R(y_0(t))$: $\forall\, t \in I_{a'}(t_0)$,

$$|y(t) - y_0(t)| = \left|\int_{t_0}^{t} f(s, x_0)\, ds\right| \leq M_0 \cdot |t - t_0| \leq M_0 \cdot a' < (1 - La')b = R$$

(since $a' < \frac{b}{M_0 + Lb} \Leftrightarrow M_0 a' + Lba' < b \Leftrightarrow M_0 a' < (1 - La')b$)

Therefore, $\exists!\, x(t) \in \overline{B_r(x_0(t))}$ such that $\Phi(x(t)) = \int_{t_0}^{t} f(s, x_0)\, ds$

i.e. $\exists!\, x(t) : I_{a'}(t_0) \to I_b(x_0)$ satisfying the integral equation.

Q2) (HW9, Q5) Prove ②.

Sol) Prove by induction on $k \geq 0$: $f \in C^k(\mathbb{R}) \Rightarrow x(t) \in C^{k+1}(I_{a'}(t_0))$

$k=0$: By ①, $\exists\, x(t) \in C[t_0-a', t_0+a']$ such that $X(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, dt$

which is therefore C^1, by the Fundamental Theorem of Calculus (FTC).

Suppose the statement holds for $k=K$, then for $k=K+1$, $\forall f \in C^{K+1}(\mathbb{R})$,

then $f \in C^K(\mathbb{R})$, hence by Inductive hypothesis $X(t) \in C^{K+1}(I_{a'}(t_0))$.

Therefore, $f(t, x(t))$ is C^{K+1}, then $X(t)$ is C^{K+2} by FTC.

$\therefore$ By Induction, $\forall k \geq 0$, $f \in C^k(\mathbb{R}) \Rightarrow x(t) \in C^{k+1}(I_{a'}(t_0))$